Self-similar solutions for the Kardar-Parisi-Zhang interface dynamic equation

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Abstract

In this article we will present a study of the well-known Kardar-Parisi-Zhang(KPZ) model. Under certain conditions we have found analytic self-similar solutions for the underlying equation. The results are strongly related to the error functions. One and two spatial dimensions are considered with different kind of self-similar Ansätze.

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I. INTRODUCTION

Growth of patterns in clusters and sodification fronts are a challenging problem from a long time. Basic knowledge of the roughness of growing crystalline facets has obvious technical applications [1]. The simplest nonlinear generalization of the ubiquitous diffusion equation is the the so called Kardar-Parisi-Zhang [2] model obtained from Langevin equation

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta(\mathbf{x}, t) \tag{1}$$

where h stands for the level of the local profile. The first term on the right hand side describes relaxation of the interface be a surface tension which prefers a smooth surface. The second term is the lowest-order nonlinear term that can appear in the surface growth equation justified with the Eden model and originates from the tendency of the surface to locally grow normal to itself and is non-equilibrium in origin. The last term is a Langevin noise to mimic the stochastic nature of any growth process and has a Gaussian distribution usually. The right hand side eventually may contain a constant term corresponding to a driving force that generates an increase of the surface and which will be discussed later. There are numerous studies available about the KPZ equation in the literature in the last fifteen years. Without completeness we mention some of them. Hwa and Frey [3] investigated the KPZ model with the help of the self-consistent mode-coupling method and with renormalisation group-theory which is an exhaustive and sophisticated method which uses Green's functions. They considered various dynamical scaling form of $C(x,t) = x^{-2\chi}C(bx, b^z t)$ for the correlation function. Lässig shows how the KPZ model can be derived and investigated with field theoretical approach [4]. In a topical review paper Kriecherbauer and Krug [5] derives the KPZ model from hydrodynamical conservation equations with a general current density relation.

One may find similar models, which may lead to similar equations to the one presented above, modeling the interface growth of bacterial colonies [6]. There is a more general interface growing model based on the so-called Kuramoto-Sivashinsky [7] equation which is basically the KPZ model with and extra $-\nabla^4 h$ term on the right hand side of (1). In the following we will not consider this model, because the self-similar Ansatz cannot be applied to this equation.

II. METHOD FOR SOLUTION

To obtain analytic solutions of the KPZ equation in the following we will use the self-similar Ansatz which can be found in [8–10]

$$T(x,t) = t^{-\alpha} f\left(\frac{x}{t^{\beta}}\right) := t^{-\alpha} f(\omega), \tag{2}$$

where $T(x,t)$ can be an arbitrary variable of a PDE and t means time and x means spatial dependence. The similarity exponents α and β are of primary physical importance since α represents the rate of decay of the magnitude $T(x,t)$, while β is the rate of spread (or contraction if $\beta < 0$) of the space distribution as time goes on. The most powerful result of this Ansatz is the fundamental or Gaussian solution of the Fourier heat conduction equation (or for Fick's diffusion equation) with $\alpha = \beta = 1/2$. (Exponents with a value of one half in another language means a typical random walk process.) These solutions are visualized on figure 1. for time-points $t_1 < t_2$. Solutions with integer exponents are called self-similar solutions of the first kind, non-integer exponents mean self-similar solutions of the second kind. The existence of self-similar solutions exclude the existence any kind of characteristic time scale as well.

Applicability of this Ansatz is quite wide and comes up in various mechanical systems [8–10], transport phenomena like heat conduction [11] or even the three dimensional Navier-Stokes equation [12].

In the following we will consider one spatial dependence of the KPZ equation. Calculating the first time and space derivative of the Ansatz (2) and pluging back to (1) (first we consider no noise term $\eta(x,t) = 0$) we get the following constrains for the exponents: $\alpha = 0$ and $\beta = 1/2$. The remaining non-linear ordinary differential equation(ODE) reads

$$\nu f''(\omega) + f'(\omega)\left[\frac{\omega}{2} + \frac{\lambda}{2} f'(\omega)\right] = 0. \tag{3}$$

The result is proportional with the logarithm of the error function

$$f(\omega) = \frac{2\nu ln\left(\frac{\lambda c_1\sqrt{\pi\nu}\ erf[\omega/(2\sqrt{\nu})] + c_2}{2\nu}\right)}{\lambda} \tag{4}$$

where erf is the error function [14] and c_1 and c_2 are integration constants. The role of c_2 is just a shift of the solution. From physical reasons ν the surface tension should be larger than zero. From analysis of the solution Eq. (4) the value of λ should be positive as well. Figure

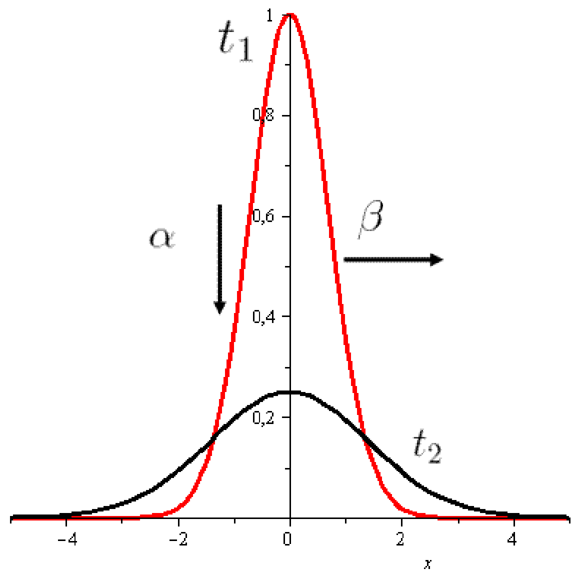

FIG. 1: A self-similar solution of Eq. (2) for $t_1 < t_2$. The presented curves are Gaussians for regular heat conduction.

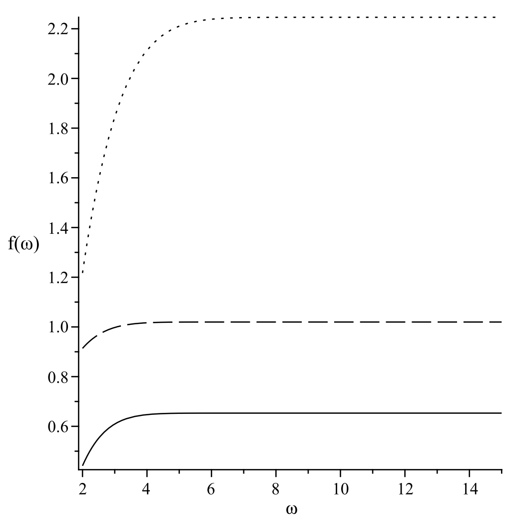

FIG. 2: The self similar solution Eq. (4) for $c_1 = c_2 = 1$. Solid line is for $\lambda = \nu = 1$, dashed line is for $\nu = 1$ and $\lambda = 2$ and the dotted line represents $\nu = 2$ and $\lambda = 1$.

2. presents two solutions with $c_1 = c_2 = 1$ and for three different λ and ν combinations. Note, that all solution has the same simple qualitative behavior, a ramp-up and a converged plateau. Figure 3. shows a three dimensional x,t dependent solution of the original equation for $c_1 = c_2 = \lambda = \nu = 1$ The function has a similar structure like Fig. 2. a ramp-up and a convergent plateau.

Let's consider some analytic noise term for the 1+1 dimensional case. From the self-similar Ansatz (2) it is obvious that the noise term $\eta(x,t)$ should be a function of $\omega = x/t^\beta$, on the other side $\eta(x,t)$ should be a distribution function as well. Therefore we first tried the following noise terms: $exp(-\omega)$ and $exp(-\omega^2)$. Unfortunately, no analytic results could be found in any closed form. For the $1/(1+\omega^2)$ (Lorenzian) noise term we got analytic results which can be expresses in sophisticated terms of Heun C special functions which we not mention here. A second analytic result was found for the $1/\omega^2$ noise term which can be evaluated with a tedious combination of the Whitakker M and Whitakker W functions which we also skip here.

If the surface growth is fed by some mechanism with a driving force, like the constant ν term that may appear in addition on the r.h.s. of Eq. 1. and which may be suggested by the work of [13], then other analytical solutions become available which can be expressed via Kummer M and Kummer U functions. If the constant driving term in its magnitude is not ν (not equal with the parameter of the Laplacean term) but a general ϵ, the solutions are even more complicated. There is no general closed form for this case where all the three parameters are free. If we fix $\lambda = \nu = 1$ than three independent cases are available. For $0 < \epsilon < 1$ the solution is like in Eq. 4. If $\epsilon = 2$ the solutions contains a complex part, however for large ($\epsilon = 10$) the solution is evaluated via the combination of hypergeometric functions. Note, that all these solutions contain two integrations constants c_1, c_2 which have non-linear dependence in the solutions.

At last we investigate the two spatial dependent cases where we generalize the self-similar Ansatz. In our former studies it came out clearly that the self-similar Ansatz of (2) can be generalised in many ways [11]/2:

$$h(x,y,t) = t^{-\alpha} f\left(\frac{F[x,y]}{t^\beta}\right) := t^{-\alpha} f(\omega), \tag{5}$$

where $F[x,y]$ can be understood as a parametrization of a 2 dimensional curve. For $F[x,y]$ we considered various functions like the most important $\sqrt{x^2+y^2} = a$ which can be interpreted

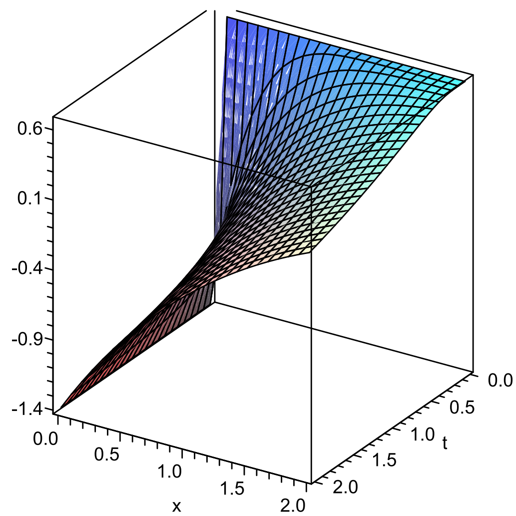

FIG. 3: The self-similar solution of the original Equation for $c_1 = c_2 = \lambda = \nu = 1$.

as the usual distance or the L^2 Euclidean norm. Unfortunately, now for the recent KPZ model only the linear case $0 = y - ax - b$ is available giving us the following ODE without any contradiction

$$\nu(a^2+1)f''(\omega) + f'(\omega)\left[\frac{\omega}{2} + \frac{\lambda}{2}f'(\omega)(a^2-1)\right] = 0. \tag{6}$$

For the exponents the $\alpha = 0$ and $\beta = 1/2$ constrains are still valid. The general analytical solution is the following:

$$f(\omega) = \frac{1}{\lambda(a^2-1)}\left(ln\left[\frac{1}{4}\frac{1}{\nu(a^2+1)}\left(\lambda^2\left\{c_1\sqrt{\pi}erf\left(\frac{1}{2}\sqrt{\frac{1}{\nu(a^2+1)}}\omega\right)a^2 - \right.\right.\right.\right.$$
$$\left.\left.\left.\left. c_1\sqrt{\pi}erf\left(\frac{1}{2}\sqrt{\frac{1}{\nu(a^2+1)}}\omega\right) + c_2\sqrt{\frac{1}{\nu(a^2+1)}}a^2 - c_2\sqrt{\frac{1}{\nu(a^2+1)}}\right\}^2\right)\right]\nu(a^2+1)\right). \tag{7}$$

Note, that this solution is very similar to the one dimensional one (4) there is a ramp-on and a plateau for $0 < \omega$ which means positive time and positive interface widths these are physical constraints of a real solution. Figure 4 presents a particular solution for the $\lambda = \nu = c_1 = c_2 = 1, a = 2$ parameters.

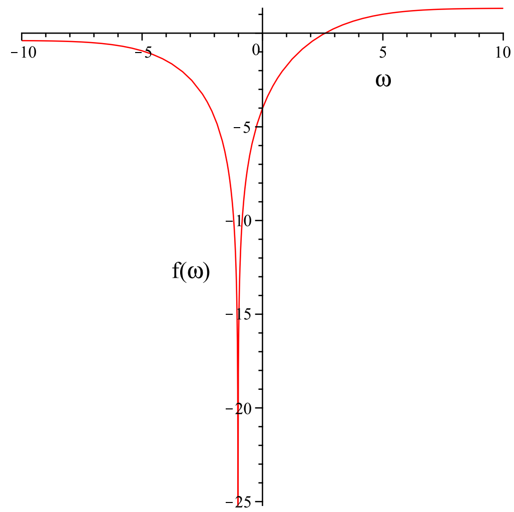

FIG. 4: Solution of Eq. 7 for $c_1 = c_2 = \lambda = \nu = 1, a = 2$.

III. CONCLUSIONS

In summary we can say, that with an appropriate change of variables applying the self-similar Ansatz one may obtain analytic solution for the KPZ equation for one or two dimension sometimes even with some noise term.

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